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Mean Interpolation of Entire Functions

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We study the characterization of an entire function from its "means," that is, a combination of the function's averages on concentric circles and its derivatives at the center. It is shown that a large class of entire functions is uniquely determined from this combination. Given a sequence $\{r_n\}$ of nonnegative radii which are restricted in growth and a sequence of complex numbers $\{\lambda_n\}$, which depends on $\{r_n\}$, a unique entire function f is found such that λ_n is the "mean" of f on the circle $|z| = r_n$, solving a mean interpolation problem. Consequently, a series representation for a given entire function is constructed from its "means."

1. INTRODUCTION AND RESULTS

Let Γ_β be the class of entire functions of growth category $(\rho, \tau) \leqslant (\beta, 0)$, i.e., the order ρ of f is less than or equal to β and if $\rho = \beta$ then the type τ is equal to 0. Let $w_n{}^k = \exp(i2\pi k/n)$, $k = 1, 2, \dots, n$, be the nth roots of unity. Given a sequence of radii $\{r_n\}$, $r_n \geqslant 0$, we consider the following "means" of an entire function f,

$$\begin{aligned} s_n(r_n, f) &= \frac{1}{n} \sum_{k=1}^{n} f(r_n w_n{}^k), && \text{if} \quad r_n > 0, \\ &= f^{(n)}(0)/n!, && \text{if} \quad r_n = 0. \end{aligned}$$

That is, if $r_n > 0$, $s_n(r_n, f)$ is the average of f at equally spaced points on the circle $|z| = r_n$, and if $r_n = 0$, $s_n(r_n, f) = a_n$, the Taylor coefficient of f at 0.

In [1], Blakley *et al.* studied the means, $s_n(r_n, \cdot)$, for functions holomorphic in the unit circle, where $0 < r_n \leqslant 1$. We obtain some analogous results for entire functions and for nonnegative radii, r_n, of restricted growth.

First, we have

THEOREM 1. *Let $f \in \Gamma_\beta$ and let $r_n > 0$ for an infinite number of n's such that $r_n = O(n^{1/\beta})$. If*

$$s_n(r_n, f) = 0, \qquad n = 1, 2, \dots, \tag{1}$$

then $f \equiv 0$.

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Thus, if $\{r_n\}$ is given as above and $f, g \in \Gamma_\beta$ such that for $n = 1, 2, \ldots$, $s_n(r_n, f) = s_n(r_n, g)$, then $s_n(r_n, f - g) = 0$ for $n = 1, 2, \ldots$, and $f \equiv g$. Therefore, certain entire functions are uniquely determined by the $s_n(r_n, \cdot)$.

As a consequence of the proof of Theorem 1, we have the following

COROLLARY. *Let f be an entire function and $r_n = 0$ for at most a finite number of n's. If $f(0) = 0$ and $s_n(r_n, f) = 0$ for $n = 1, 2, \ldots$, then $f \equiv 0$.*

None of the conditions in (1) can be left out, as seen in

THEOREM 2. *Let $r_n \geq 0$. For each positive integer m there is a unique polynomial p_m of degree m, leading coefficient equal to 1, and $p_m(0) = 0$ such that, for $n = 1, 2, \ldots$,*

$$\begin{aligned} s_n(r_n, p_m) &= r_n{}^n \delta_{n,m}, & &\text{if} \quad r_m > 0, \\ &= \delta_{n,m}, & &\text{if} \quad r_m = 0. \end{aligned} \tag{2}$$

It will be shown that if all $r_n = 0$ then $p_m \equiv z^m$, as would be expected. Let

$$\begin{aligned} \hat{s}_n(r_n, f) &= s_n(r_n, f)/r_n{}^n, & &\text{if} \quad r_n > 0, \\ &= f^{(n)}(0)/n!, & &\text{if} \quad r_n = 0. \end{aligned} \tag{3}$$

Given a sequence of nonnegative real numbers $\{r_n\}$, (the "mean" interpolation radii), and a sequence of complex numbers $\{\lambda_n\}$, (the mean data), is there a unique function f such that $\hat{s}_n(r_n, f) = \lambda_n$, for all n? We have the following answer.

THEOREM 3. *Let $r_n = O(n^{1/\beta})$, $\beta > 0$, and let $\{\lambda_n\}$ be any sequence of complex numbers satisfying*

$$\lim_{n \to \infty} n \mid \lambda_n \mid^{\beta/n} = 0. \tag{4}$$

Then the polynomial series

$$\sum_{n=1}^{\infty} \lambda_n p_n(z) \tag{5}$$

converges uniformly on every compact set of the complex plane to an entire function f in Γ_β such that $\hat{s}_n(r_n, f) = \lambda_n$, $n = 1, 2, \ldots$. Furthermore, f is the only function in Γ_β which satisfies this mean interpolation property.

The following theorem will allow us to reconstruct an entire function f from the $s_n(r_n, f)$, where the λ_n of (4) will be replaced by

$$\begin{aligned} q_n(r_n, f) &= (s_n(r_n, f) - f(0))/r_n^n, &\quad \text{if} \quad r_n > 0, \\ &= s_n(r_n, f), &\quad \text{if} \quad r_n = 0. \end{aligned} \tag{6}$$

Note, $q_n(r_n, f) = \hat{s}_n(r_n, f)$, if $f(0) = 0$.

Finally, letting Λ_β, a subset of Γ_β, be the set all entire functions of order strictly less that β, we have

THEOREM 4. *Let $r_n \geqslant 0$ and $r_n = O(n^{1/\beta})$. Every function f in Λ_β can be represented by the polynomial series*

$$f(z) = f(0) + \sum_{n=1}^{\infty} q_n(r_n, f)\, p_n(z), \tag{7}$$

where the p_n are given in Theorem 2.

2. UNIQUENESS RESULTS

Let ρ be the order and τ be the type of a function f. It is known [cf. [2]] that

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} = \rho, \tag{8}$$

and

$$\limsup_{n\to\infty} n\, |a_n|^{\rho/n} = e\tau\rho, \qquad \text{if} \quad 0 < \rho < \infty. \tag{9}$$

We will need the following lemma which is a consequence of (8) and (9).

LEMMA 1. *Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be of growth category (ρ, τ). Then $(\rho, \tau) \leqslant (\beta, 0)$ for some $\beta > 0$ if and only if*

$$\lim_{n\to\infty} n\, |a_n|^{\beta/n} = 0. \tag{10}$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If $r_n > 0$, then

$$s_n(r_n, f) = \sum_{k=0}^{\infty} a_k r_n^k \left(\frac{1}{n} \sum_{j=1}^{n} w_n^{jk}\right) = \sum_{k=0}^{\infty} a_{nk} r_n^{nk}.$$

If $s_n = 0$, we have

$$a_0/r_n + \sum_{k=1}^{\infty} a_{nk} r_n^{n(k-1)} = 0, \qquad \text{for} \quad r_n > 0. \tag{11}$$

It will be necessary in the proof of Theorem 1 to show that $f(0) = a_0 = 0$. To do this we have

LEMMA 2. *Let $f \in \Gamma_\beta$, $r_n = O(n^{1/\beta})$ and $\{r_{n_j}\}$ be a subsequence such that $r_{n_j} > 0$ for each j. If $s_{n_j}(r_{n_j}, f) = 0$, then $f(0) = 0$.*

Proof. By hypothesis and Eq. (11), we have

$$a_0 = -\sum_{k=1}^{\infty} a_{n_jk} r_{n_j}^{n_jk}, \qquad j = 1, 2, \ldots .$$

Thus,

$$|a_0| \leq \sum_{k=1}^{\infty} |a_{n_jk}| \, r_{n_j}^{n_jk}, \tag{12}$$

for each j.

In order to complete the proof of Lemma 2, let $c > 0$ such that $r_n < cn^{1/\beta}$ for all n and let $0 < \epsilon < c^{-\beta}$. Since $f \in \Gamma_\beta$, we have by Lemma 1, that $|a_n| < (\epsilon/n)^{n/\beta}$ for all large n and Eq. (12) becomes

$$\begin{aligned} |a_0| &< \sum_{k=1}^{\infty} \left(\frac{\epsilon}{n_jk}\right)^{n_jk/\beta} \cdot (c^\beta n_j)^{n_jk/\beta} \\ &\leq \sum_{k=1}^{\infty} (\epsilon c^\beta)^{n_jk}. \end{aligned}$$

The series is convergent for each n_j since $\epsilon c^\beta < 1$. Thus as $j \to \infty$ the series tends to zero. Therefore, $f(0) = a_0 = 0$, which completes the proof of Lemma 2.

Proof of Theorem 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then by (11) and Lemma 2

$$\sum_{k=1}^{\infty} a_{nk} r_n^{n(k-1)} = 0, \qquad \text{if} \quad r_n > 0. \tag{13}$$

Using the definition of s_n for $r_n = 0$ and the fact that each $s_n = 0$, we have

$$a_n = 0, \qquad \text{if} \quad r_n = 0. \tag{14}$$

Equations (13) and (14) form an infinite homogeneous system of equations. It is, therefore, necessary and sufficient to prove this system has only the trivial solution. Let $B = (b_{j,k})$ be the infinite coefficient matrix given by

$$\begin{aligned} b_{j,k} &= r_j^{k-j}, \qquad &\text{if } j \mid k, \\ &= 0, \qquad &\text{if } j \nmid k \end{aligned} \tag{15}$$

where $r_j^0 = 1$, even if $r_j = 0$. Equations (13) and (14) can be written as the matrix equation $BA^T = O$, where $A = (a_1, a_2, \ldots)$.

Let $B_N = (b_{j,k})_{1 \leqslant j,k \leqslant N}$, $N = 1, 2, \ldots$, be the truncated $N \times N$ matrices. Since $\det(B_N) = 1$, for each N, there exists an inverse G_N of B_N for each N, which is a truncation of the infinite matrix

$$G = (g_j(k)) = \begin{bmatrix} g_1(1) & g_1(2) & \cdots \\ g_2(1) & g_2(2) & \cdots \\ \vdots & \vdots & \end{bmatrix}.$$

In fact $G_N B_N = I_N$, where I_N is the $N \times N$ identity matrix and so

$$\sum_{k=1}^{N} g_j(k)\, b_{k,n} = \delta_{j,n}, \qquad 1 \leqslant j, \quad n \leqslant N,$$

where $\delta_{j,n}$ is the Kronecker delta. Using (15), we have

$$\sum_{k \mid n} g_j(k)\, r_k^{n-k} = \delta_{j,n}, \tag{16}$$

which is independent of N.

By induction it was shown in [1] that

$$g_j(n) = 0, \qquad \text{if} \quad j \nmid n \tag{17}$$

and

$$g_j(j) = 1, \qquad j = 1, 2, \ldots,$$

and it follows from (16) that

$$g_j(n) = -\sum_{\substack{k \mid n \\ j \leqslant k < n}} g_j(k)\, r_j^{n-j}, \qquad j \mid n, \quad j < n. \tag{18}$$

Let h be the function defined recursively on the set of positive integers by

$$h(1) = 1,$$
$$h(n) = \sum_{\substack{l \mid n \\ l < n}} h(l), \qquad \text{if} \quad n > 1.$$

Later, we will use the following lemma from [1].

LEMMA 3. *Let $h(n)$ be defined as above, then*

$$h(n) \leqslant 2^{(\log n / \log 2)^2}, \qquad n = 1, 2, \ldots .$$

Letting $\sigma_n = \max_{1 \leqslant k \leqslant n} \{r_k\}$, we have the following bound on $g_j(k)$.

LEMMA 4. *For each j and k*

$$|g_j(k)| \leqslant h(k)\,\sigma_k^{k-j},$$

where $\sigma_k^0 = 1$, if $\sigma_k = 0$.

Proof. Since $h(k) \geqslant 1$ and $\sigma_k \geqslant 0$,

$$g_j(k) = 0 \leqslant h(k)\cdot\sigma_k^{k-j} \qquad \text{if} \quad j \nmid k,$$

and

$$g_j(j) = 1 \leqslant h(j) = h(j)\,\sigma_j^{j-j}.$$

Assume that for each j, $j \mid k$, Lemma 4 is true for each d, $1 \leqslant d < k$. Then, by (18) and the fact that $\sigma_k \leqslant \sigma_{k+1}$, we have

$$\begin{aligned}|g_j(k)| &\leqslant \sum_{\substack{d \mid k\\ d<k}} |g_j(d)|\,r_d^{k-d}\\ &\leqslant \sum_{\substack{d \mid k\\ d<k}} (h(d)\,\sigma_d^{d-j})\,\sigma_d^{k-d}\\ &\leqslant \sigma_k^{k-j} \sum_{\substack{d \mid k\\ d<k}} h(d) = \sigma_k^{k-j}h(k),\end{aligned}$$

which completes the proof.

We are now ready to complete the proof of Theorem 1. By matrix multiplication [cf. [1]] we have for each j,

$$|a_j| \leqslant \sum_{k=N+1}^{\infty} |a_k c_k|, \qquad N = j+1, j+2, \ldots, \tag{19}$$

where

$$c_k = \sum_{\substack{d \mid k\\ d\leqslant k}} g_j(d)\,r_d^{k-d}.$$

We wish to show the series in (19) is convergent, for then the right-hand side would go to zero as $N \to \infty$, implying $a_j = 0$.

From the proof of Lemma 4 and the fact that $k > N$, it follows that $|c_k| \leqslant \sigma_k^{k-j}\cdot h(k)$. Since $r_n = O(n^{1/\beta})$, then there is a constant $c > 0$, such that $\sigma_n \leqslant cn^{1/\beta}$ for all n.

Let $0 < \epsilon < 1/c$. By Lemma 1, $|a_k|^{1/k} \leqslant \epsilon/k^{1/\beta}$ and

$$\begin{aligned}|a_k c_k|^{1/k} &\leqslant \sigma^{1-j/k}(h(k))^{1/k}\cdot \epsilon/k^{1/\beta}\\ &\leqslant (\epsilon c)[h(k)/(ck^{1/\beta})^j]^{1/\beta}\end{aligned}$$

for all large k. According to Lemma 3, it follows that

$$\lim_{k\to\infty} \sup[h(k)/(ck^{1/\beta})^j]^{1/k} = a < 1.$$

Thus,

$$\lim_{k\to\infty} \sup | a_k c_k |^{1/k} \leqslant \epsilon c < 1$$

and hence $\sum_{k=N+1}^{\infty} | a_k c_k |$ converges. Taking $N \to \infty$ in (19), we obtain $a_j = 0$ for each $j = 1, 2,\ldots$. Therefore $f(z) \equiv a_0 = 0$, which completes the proof of Theorem 1.

Proof of Corollary. Since $f(0) = 0$ we may write $f(z) = \sum_{k=1}^{\infty} a_k z^k$. There exists a positive integer N, such that $r_N > 0$, and $0 = r_{N+1} = r_{N+2} = \cdots$. Thus $s_n(r_n, f) = a_n = 0$ for $n = N+1, N+2,\ldots$, and $f(z) = \sum_{k=1}^{N} a_k z^k$. From Eqs. (13) and (14) of Theorem 1, we obtain

$$\sum_{k=1}^{[N/n]} a_{nk} r_n^{n(k-1)} = 0, \qquad \text{if} \quad r_n > 0$$

and

$$a_n = 0, \qquad \text{if} \quad r_n = 0.$$

which, for $1 \leqslant n \leqslant N$, forms an $N \times N$ homogeneous system of linear equations. This system is represented by the matrix equation

$$B_N A^T = O_{N\times N},$$

where $A = (a_1,\ldots, a_N)$ and B_N is the truncated matrix of Theorem 1, which is nonsingular. Hence, the only solution is $A = 0$ and, therefore, $f(z) \equiv 0$.

3. Representation by Polynomial Series

We are now ready to present the

Proof of Theorem 2. Let $p_m(z) = a_m z^m + \cdots + a_1 z$, and $n > m$. Then $n \nmid k$, $k = 1,\ldots, m$ and hence

$$s_n(r_n, p_m) = \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{m} a_k w^{jk}$$

$$= \sum_{k=1}^{m} a_k \frac{1}{n} \sum_{j=1}^{n} (w^k)^j = 0$$

for $r_n > 0$. If $r_n = 0$, then $s_n(r_n, p_m) = p_m^{(n)}(0) = 0$, since $m < n$.

In order to determine p_m, we need to consider Eqs. (2) only for $n = 1, \ldots, m$. From (2) and (11) we have

$$\sum_{k=1}^{[m/n]} a_{nk} r_n^{n(k-1)} = 0, \qquad \text{if} \quad r_n > 0, \quad n < m,$$

$$a_n = 0, \qquad \text{if} \quad r_n = 0,$$

and

$$a_m = 1, \qquad \text{if} \qquad r_m > 0, \qquad \text{or} \qquad r_m = 0.$$

In all cases, the coefficients $a_1, \ldots, a_m$ of p_m are uniquely determined by the nonhomogeneous system

$$B_m \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where B_m is the truncated characteristc matrix in the proof of Theorem 1 with inverse $G_m = (g_j(k))_{1 \leqslant j, k \leqslant m}$. Thus,

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = G_m \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{20}$$

Since $g_m(m) = 1$, $a_m = 1$ and this completes the proof.

In fact, we can derive p_m explicity. From (17) and (20), $a_k = g_k(m) = 0$ if $k \nmid m$. Hence, p_m is given by

$$p_m(z) = \sum_{k|m} g_k(m)\, z^k. \tag{21}$$

If all r_n are zero we obtain $p_m(z) = z^m$. This is true because $g_k(m) = 0$, if $k < m$ and $r_k = 0$. Indeed, from (18) $g_k(2k) = -g_k(k)\, r_k{}^k = 0$. Assume $g_k(d) = 0$, for each d, $k < d < m$. Again from (17)

$$g_k(m) = - \sum_{\substack{d|m \\ k<d<m}} g_k(d)\, r_d^{m-d} = 0.$$

We are now ready to prove Theorem 3 on interpolation.

Proof of Theorem 3. First we prove the convergence of the polynomial series (5). Let $|z| \leqslant r$. From (21) and Lemma 3, it follows that

$$\begin{aligned} |p_n(z)| &\leqslant \sum_{k|n} |g_k(n)|\,|z|^k \\ &\leqslant \sum_{k|n} \sigma_n^{n-k} h(n)\, r^k \\ &\leqslant nh(n)[\max\{\sigma_n, r\}]^n. \end{aligned}$$

If $r_n \leqslant M$, for all n, then $\max\{\sigma_n, r\} < c_r$ for some constant c_r, independent of z and n. If $|z| \leqslant r$, then

$$|\lambda_n p_n(z)|^{1/n} \leqslant c_r(nh(n))^{1/n} |\lambda_n|^{1/n}.$$

Since $h(n) < 2^{(\log n/\log 2)^2}$, we have $\lim_{n\to\infty} \sup[nh(n)]^{1/n} = a \leqslant 1$, and since $\lim_{n\to\infty} |\lambda_n|^{1/n} = 0$, it follows that

$$\lim_{n\to\infty} |\lambda_n p_n(z)|^{1/n} = 0.$$

Thus, the series $\sum_{n=1}^{\infty} \lambda_n p_n(z)$ converges uniformly on every compact set of the complex plane.

Suppose, however, $r_n \to \infty$ as $n \to \infty$, then for all large n, $\max\{\sigma_n, r\} = \sigma_n$ and if $|z| \leqslant r$, then

$$|\lambda_n p_n(z)|^{1/n} \leqslant (nh(n))^{1/n}\sigma_n |\lambda_n|^{1/n}.$$

By the hypotheses, there exist $d > 0$ and $\epsilon > 0$ such that for all large n $\sigma_n \leqslant dn^{1/\beta}$ and $|\lambda_n|^{1/n} < \epsilon/n^{1/\beta}$. If $|z| \leqslant r$, then

$$|\lambda_n p_n(z)|^{1/n} \leqslant \epsilon d(nh(n))^{1/n}$$

for all large n and so

$$\limsup_{n\to\infty} |\lambda_n p_n(z)|^{1/n} \leqslant \epsilon d < 1$$

uniformly for $|z| \leqslant r$. Therefore, the series $\sum_{n=1}^{\infty} \lambda_n p_n(z)$ converges to some entire function f, and we may write $f(z) = \sum_{n=1}^{\infty} \lambda_n p_n(z)$. Since $p_n(0) = 0$ for all n, $f(0) = 0$.

Write $f(z) = \sum_{k=1}^{\infty} a_k z^k$. In order to show that $f \in \Gamma_\beta$, it must be shown that $\lim_{k\to\infty} k\,|a_k|^{\beta/k} = 0$, according to Lemma 1. Now by convergence,

$$\begin{aligned} \sum_{k=1}^{\infty} a_k z^k &= \sum_{n=1}^{\infty} \lambda_n p_n(z) \\ &= \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k|n} g_k(n)\, z^k\right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \lambda_{kn}\, g_k(kn)\right) z^k. \end{aligned}$$

Equating coefficients and noting that $g_k(k) = 1$, we have

$$a_k = \sum_{n=1}^{\infty} \lambda_{kn} g_k(kn) = \lambda_k + \sum_{n=2}^{\infty} \lambda_{kn} g_k(kn).$$

Recall that $\sigma_n \leqslant dn^{1/\beta}$ for all n. For any $\epsilon > 0$, $\epsilon < 1/(2d)$, we have $| \lambda_n | < (\epsilon/n^{1/\beta})^n$ for all large n. Thus, for large n,

$$\begin{aligned} | a_k | &\leqslant | \lambda_k | + \sum_{n=2}^{\infty} | \lambda_{kn} | \, | g_k(kn)| \\ &\leqslant | \lambda_k | + \sum_{n=2}^{\infty} | \lambda_{kn} | \, \sigma_{kn}^{kn-k} h(kn) \\ &\leqslant \frac{\epsilon^k}{k^{k/\beta}} + \sum_{n=2}^{\infty} \frac{\epsilon^{kn}}{kn^{kn/\beta}} [d(kn)^{1/\beta}]^{kn-k} h(kn) \\ &\leqslant \frac{\epsilon^k}{k^{k/\beta}} \left[1 + \sum_{n=2}^{\infty} (\epsilon d)^{k(n-1)} h(kn) \right]. \end{aligned}$$

Now $h(kn) \leqslant 2^{(\log kn/\log 2)^2} < 2^{k(n-1)}$, for large n, and so

$$| a_k | \leqslant \frac{\epsilon^k}{k^{k/\beta}} \sum_{n=1}^{\infty} (2\epsilon d)^{k(n-1)}.$$

The series in the above inequality converges. Thus as $k \to \infty$, the series tends to zero, then $| a_k | < c\epsilon^k/k^{k/\beta}$ for some constant c and all large k. Since ϵ is arbitrary, it follows that $\lim_{k\to\infty} k \, | a_k |^{\beta/k} = 0$. Therefore, f is of growth category $(\rho, \tau) \leqslant (\beta, 0)$ and so $f \in \Gamma_\beta$.

By Theorem 2 and the definition of $\hat{s}_n(r_n, f)$ in (3),

$$s_n(r_n, f) = \sum_{m=1}^{\infty} \lambda_m \hat{s}_n(r_n, p_m) = \lambda_n$$

for each $n = 1, 2, \ldots$. Furthermore, if $g \in \Gamma_\beta$ and $\hat{s}_n(r_n, g) = \lambda_n$ for $n = 1, 2, \ldots$, then $\hat{s}_n(r_n, f - g) = 0$ and, hence, $s_n(r_n, f - g) = 0$. By Theorem 1, $f = g$, which completes the proof of Theorem 3.

Proof of Theorem 4. We will show that any $f \in \Gamma_\beta$ is given by (7). First let

$$g(z) = f(0) + \sum_{n=1}^{\infty} \lambda_n p_n(z),$$

where $\lambda_n = q_n(r_n, f)$ (see (6)). If it can be shown that

$$\lim_{n\to\infty} n \mid \lambda_n \mid^{\beta/n} = 0, \tag{22}$$

then, according to Theorem 3, we will have $g \in \Gamma_\beta$. If $r_m = 0$,

$$\begin{aligned} s_m(r_m, g) &= s_m(r_m, f(0)) + \sum_{n=1}^{\infty} \lambda_n s_m(r_m, p_n) \\ &= 0 + \lambda_m = q_m(r_m, f) \\ &= s_m(r_m, f). \end{aligned}$$

If $r_m > 0$, then

$$\begin{aligned} s_m(r_m, g) &= s_m(r_m, f(0)) + r_m{}^m \lambda_m \\ &= f(0) + r_m{}^m \frac{[s_m(r_m, f) - f(0)]}{r_m{}^m} \\ &= s_m(r_m, f). \end{aligned}$$

Thus $s_n(r_n, f) = s_n(r_n, g)$, $n = 1, 2, \ldots$. Since $f \in \Lambda_\beta \subset \Gamma_\beta$ and $g \in \Gamma_\beta$, then, by Theorem 1, $f \equiv g$.

We now prove (22). Write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Since $f \in \Lambda_\beta$, then $\lim_{n\to\infty} n \mid a_n \mid^{\beta/n} = 0$. If $r_n = 0$, $\lambda_n = q_n(r_n, f) = s_n(r_n, f) = f^{(n)}(0)/n! = a_n$ and (22) follows immediately. If $r_n > 0$, then by the definition of $q_n(r_n, f)$

$$\begin{aligned} \lambda_n &= q_n(r_n, f) = (s_n(r_n, f) - f(0)/r_n) \\ &= a_n + \sum_{k=2}^{\infty} a_{nk} r_n^{nk-n}. \end{aligned}$$

Let $\epsilon > 0$ be given such that $\epsilon d < 1$, where $r_n < dn^{1/\beta}$ for all n. We have for large n,

$$\begin{aligned} \mid \lambda_n \mid &\leqslant \mid a_n \mid + \sum_{k=2}^{\infty} \mid a_{nk} \mid r_n^{nk-n} \\ &\leqslant \frac{\epsilon^n}{n^{n/\beta}} + \sum_{k=2}^{\infty} \frac{\epsilon^{nk}}{(nk)^{nk/\beta}} \cdot d^{nk-n} n^{(nk-n)/\beta} \\ &\leqslant \frac{\epsilon^n}{n^{n/\beta}} \left(1 + \sum_{k=2}^{\infty} (\epsilon d)^{nk}\right). \end{aligned}$$

The geometric series converges, and, thus, tends to zero as $n \to \infty$. Therefore, $\lim_{n\to\infty} n \mid \lambda_n \mid^{n/\beta} = 0$, which completes the last proof.

FINAL REMARKS

For a given sequence of radii r_n, $r_n = O(n^{1/\beta})$, we can characterize large classes of entire functions from their "means," $s_n(r_n, \cdot)$. However, we would like to know if Γ_β in Theorems 1 and 3 and Λ_β in Theorem 4 are the largest classes possible.

REFERENCES

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